

A geometric characterization of Poisson type distributions

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Abstract: Tempered distributions on \mathbb{R}^n that have the Poisson structure are characterized in terms of geometry of its support and spectrum

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1 Introduction

It is known in crystallography that the diffracted wave function is the Fourier transform of the electronic density function. The old crystallographic wisdom was whenever one sees isolated Bragg's peaks in the Fourier spectrum, one must have a periodic structure, i.e. a crystal. The term "quasicrystal" appeared after the discovery of "a new principle of packing of molecules" [4]. This term is used in mathematics as the name of distributions that has closed discrete support and diffraction spectrum (support of the Fourier transform) and some additional properties. A survey of the theory of quasicrystals is given in Lagarias [11]. A set $\Lambda \subset \mathbb{R}^n$ is called *uniformly discrete* (u.d.) if

$$d(\Lambda) = \inf |p - p'| > 0, \quad p \neq p' \in \Lambda.$$

The following important result was stated by N. Lev and A. Olevskii [14], conjectured by Lagarias [11].

Theorem 1 *If $\mu = \sum_{\Lambda} \mu(p) \delta_p$, $\mu(p) > 0$ is a positive measure on \mathbb{R}^n supported by a u.d. set Λ such that the Fourier transform is also a measure $\hat{\mu} = \sum_{\Sigma} \hat{\mu}(\sigma) \delta_{\sigma}$, $\hat{\mu}(\sigma) \neq 0$ supported on a u.d. set Σ , then Λ is contained in the finite union of shifts of a full rank lattice L in \mathbb{R}^n . The same is true for Σ and the dual lattice L^* . In the case $n = 1$ these conclusions hold without the positivity assumption.*

Previous results in this direction were obtained by Córdoba [5], Kolountzakis-Lagarias [9], Meyer [3] and Favorov [15]. S. Favorov has shown that a u.d. set Λ must be the union of shifts of a finite number of (may be not parallel) lattices provided that Σ is countable and the set $\{|\mu_p|, p \in \Lambda\}$ is finite. These results

are based on Helson-Cohen's theory of idempotent measures on locally compact abelian groups. This theory depends on the choice axiom (or Tychonoff's theorem). Constructive proofs of these results are desirable. The arguments of [14] are constructive and incorporate the concept of Meyer sets [3], [6].

A full rank lattice in $E = \mathbb{R}^n$ is the set $L = g\mathbb{Z}^n$, where g is a nonsingular $n \times n$ matrix. It is the additive subgroup of E generated by columns of g (called generators of the lattice). The lattice $L^* = h\mathbb{Z}^n$, $h = (g^t)^{-1}$ in the dual space E^* is called dual. A L -crystal is the set $\Lambda = \cup_1^K L + q_k$, where $q_1, \dots, q_K \in E$. S. Favorov [15] gave an example of a measure μ on \mathbb{R}^2 whose support Λ is the u.d. union of two non parallel crystals Λ_1 and Λ_2 and the same is true for the spectrum. Favorov's measure μ has no Poisson structure. The purpose of this paper is to prove

Theorem 2 *If a tempered distribution $t \neq 0$ on \mathbb{R}^n is supported by a set Λ such that $\Lambda - \Lambda$ is u.d. and the spectrum of t is supported by a u.d. set Σ , then Λ is the L -crystal with a lattice L and Σ is the L^* -crystal with the dual lattice L^* .*

Note that any crystal Γ is a u.d. set and is true for the set $\Gamma - \Gamma$. Therefore the condition $\Lambda - \Lambda$ and Σ are u.d. in Theorem 2 is necessary for the conclusion. In view of symmetry of support and spectrum of a distribution, the condition Λ and $\Sigma - \Sigma$ are u.d. is also a characterization of crystals. We shall also show that any distribution t as above has Poisson structure (6) (Poisson's comb).

For measures $t = \mu$ these results were stated by Lev and Olevskii [12], see in [14] for father results. Our proof is based on the result of [14]. The assumption that $\Lambda - \Lambda$ is u.d. implies that Λ is itself a u.d. set. This later condition is not sufficient, which follows from Favorov's example [15]. See [10] for a survey of properties of sets Λ such that $\Lambda - \Lambda$ is u.d.

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2 Preliminaries

Let $x = (x_1, \dots, x_n)$ be a system of linear coordinates in $E \cong \mathbb{R}^n$. A Schwartz (test) function on E is a smooth function φ such that $P(x) D^i \varphi(x)$ is bounded for any polynomial P and any partial derivative $D^i = \partial_{x_1}^{i_1} \dots \partial_{x_n}^{i_n}$, $i = (i_1, \dots, i_n) \in \mathbb{Z}^n$. The vector space $S(E)$ of all Schwartz functions is called the Schwartz space and is supplied with the natural topology. A tempered distribution t on E is a linear continuous functional on the Schwartz space see more detail in [2]. For a point $p \in E$, the functional $\delta_p(\varphi) = \varphi(p)$ is called delta distribution at p . Any partial derivative of δ_p is a tempered distribution. A tempered function is a linear continuous functional on the Schwartz space of densities φdx , where φ is a test function and $dx = dx_1 \wedge \dots \wedge dx_n$. For a tempered function f , the product $f dx$ is a tempered distribution. The Fourier transform $\hat{t} = F_{x \rightarrow \xi}(t)$ of a tempered distribution t is a tempered function on the dual space E^* (L.

Schwartz's theorem, see [2]). It is defined by

$$\hat{t}(\rho) = t(F(\rho)), \quad F(\rho) = \int \exp(-j \langle \xi, x \rangle) \psi(\xi) d\xi,$$

where $\rho = \psi d\xi$, ψ is a test function on E^* , ξ_1, \dots, ξ_n are dual coordinates in E^* and $j = 2\pi i$. Let $B(\rho)$ be the open ball in E or in E^* of radius ρ with the center at the origin.

Proposition 3 *Any tempered distribution t on E supported on a u.d. set Λ is uniquely represented in the form*

$$t = \sum_{p \in \Lambda} t_p(D) \delta_p, \quad (1)$$

where $t_p(D)$ is a differential operator on E and for any test function α on E ,

$$t(\alpha) = \sum_{p \in \Lambda} \sum_{|i| \leq m(p)} \delta_p [t_p(-D) \alpha(x)].$$

The sum (1) is a tempered distribution if and only if $\text{ord } t \doteq \sup \text{ord } t_p < \infty$, and

$$\sum_{p \in \Lambda} (|p| + 1)^{-m} \|t_p(D)\| < \infty \quad (2)$$

for a constant k , where

$$\|t_p(D)\| \doteq \max_i |t_{p,i}|, \quad t_p(D) = \sum t_{p,i} D^i, \quad t_{p,i} \in \mathbb{C}.$$

Proof. By [2], t satisfies

$$|t(\varphi)| \leq C \sup_{|i| \leq k} \sup_{x \in E} (1 + |x|)^m |D^i \varphi(x)| \quad (3)$$

for some k, m and $C > 0$. For any point $p \in \Lambda$, we choose a neighborhood U_p of p such that $U_p \cap U_q = \emptyset$ for $p \neq q$. Let e_p be a test function on E whose support is contained in U_p that is equal to 1 on a neighborhood of p . We have

$$t(\varphi) = \sum_{p \in \Lambda} e_p t(\varphi),$$

since $\varphi - \sum e_p \varphi$ vanishes on a neighborhood of Λ . The distribution $e_p t$ is supported by p . By L. Schwartz's theorem, $e_p t = t_p(D) \delta_p$ for a differential operator $t_p(D)$ see [2], Theorem 24.6. Inequality (3) implies that $\text{ord } t \leq k$ and (2). \square

3 Starting the proof

According to [1] and [7], the convolution $t * t^*$ is called the *autocorrelation* of the "signal" t , where

$$t^*(x) = \bar{t}(-x) = \sum t_i(-x) (-D)^i.$$

Following [14], we define the regularized autocorrelation $A_\varphi \doteq \varphi t * t^*$ for an arbitrary tempered t taking for φ an arbitrary test function on E .

Lemma 4 Σ is a Γ -crystal with a lattice Γ .

Proof of Lemma. We have

$$A_\varphi = \sum_{p \in \Lambda} \varphi(p) t_p(D) \delta_p * \sum_{q \in \Lambda} \bar{t}_q(-D) \delta_{-q}.$$

We integrate by parts near each point $q \in \Lambda$ and apply equation $\delta_p * \delta_{-q} = \delta_{p-q}$, which yields

$$A_\varphi = \sum_p \sum_q \bar{t}_q(-D) \varphi(q) t_p(D) \delta_{p-q}(x) = \sum_{r \in \Lambda - \Lambda} a_{\varphi, r} \delta_r(x),$$

where the series

$$a \doteq \sum_{q, q+r \in \Lambda} \bar{t}_q(-D) \varphi(q) t_{q+r}(D)$$

converges for any $r \in \Lambda - \Lambda$, since by Proposition 3, the norm $\|t_q \bar{t}_{q+r}\|$ has at most polynomial growth as $q \rightarrow \infty$. This implies that distribution

$$A_\varphi = \sum_{r \in \Lambda - \Lambda} a_{\varphi, r} \delta_r(x)$$

is supported by the u.d. set $\Lambda - \Lambda$. Farther, we have $\overline{\bar{T}}$

$$F_{x \mapsto \xi}(A_\varphi) = \widehat{\varphi} \bar{t} \cdot \bar{t} = (\hat{\varphi} * \bar{t}) \cdot \bar{t}.$$

By Proposition 3, the number $k \doteq \text{ord } \hat{t}$ is finite. For an arbitrary vector $i \in \mathbb{Z}_+^n$, $|i| \leq m$, we choose a test function φ_i such that $\text{supp } \hat{\varphi}_i \subset B(d(\Sigma))$ and

$$\hat{\varphi}_i(\xi) = \frac{(-1)^{|i|}}{i!} \xi^i$$

on $B(d(\Sigma/2))$, where $i! = i_1! \dots i_n!$. Taking $\varphi = \varphi_i$ we obtain for any test function α on E^* ,

$$\begin{aligned} \left[(\hat{\varphi}_i * \bar{t}) \cdot \bar{t} \right](\alpha) &= \bar{t} \left[(\hat{\varphi}_i * \bar{t})(\alpha) \right] = \bar{t}(\xi, -D_\xi) (\bar{t}(\eta, D_\eta) (\hat{\varphi}_i(\xi - \eta)) \alpha(\xi)) \\ &= (-1)^{|i|} \sum_{\sigma \in \Sigma} \bar{t}_\sigma(-D_\xi) \left(\hat{t}_\sigma(D_\eta) \frac{(\xi - \eta)^i}{i!} \alpha(\xi) \right) \Big|_{\eta = \xi = \sigma}. \end{aligned}$$

For any $\sigma \in \Sigma$, we have

$$\begin{aligned} &(-1)^{|i|} \bar{t}_\sigma(-D_\xi) \left(\hat{t}_\sigma(D_\eta) \frac{(\xi - \eta)^i}{i!} \alpha(\xi) \right) \Big|_{\eta = \xi = \sigma} \\ &= (-1)^{|i|} \sum_j \bar{t}_{\sigma, j}(-D_\xi) \left(\sum_k \bar{t}_{\sigma, k}(D_\eta) \frac{(\xi - \eta)^i}{i!} \alpha(\xi) \right) \Big|_{\eta = \xi = \sigma} \\ &= \sum_{j+k=i} \bar{t}_{\sigma, j} \hat{t}_{\sigma, k} \alpha(\sigma), \end{aligned}$$

where we write

$$\hat{t}_\sigma(D_\xi) = \sum_{|j| \leq m} \hat{t}_{\sigma j} D_\xi^j.$$

Taking the sum over i with weights λ^i , $\lambda \in \mathbb{R}^n$, we get

$$\begin{aligned} \sum_{|i| \leq 2m} \lambda^i \left[(\hat{\varphi}_i * \hat{t}) \cdot \bar{\hat{t}} \right] (\alpha) &= \sum_{\sigma} \sum_i \lambda^i \sum_{j+k=i} \bar{\hat{t}}_{\sigma j} \hat{t}_{\sigma k} \alpha(\sigma) \\ &= \sum_{\sigma} \sum_j \sum_k \lambda^j \bar{\hat{t}}_{\sigma j} \sum_k \lambda^k \hat{t}_{\sigma k} \alpha(\sigma) = \sum_{\sigma} \left| \sum_k \lambda^k \hat{t}_{\sigma k} \right|^2 \alpha(\sigma). \end{aligned}$$

It follows that functional

$$F_{x \rightarrow \xi}(A_\varphi) = \sum_i F(\varphi_i t * t^*) = \sum_{\sigma \in \Sigma} \left| \sum_k \lambda^k \hat{t}_{\sigma k} \right|^2 \alpha(\sigma)$$

is a non negative measure supported on Σ , where

$$\varphi = \sum \lambda^i \varphi_i.$$

We shall obtain a strictly positive measure, if choose real λ in such a way that the sum $\sum \lambda^k \hat{t}_{\sigma k}$ does not vanish for each point $\sigma \in \Sigma$. It can be done, since the zero set Z_σ of this polynomial in \mathbb{R}^n is nowhere dense for any $\sigma \in \Sigma$, since $t_\sigma \neq 0$ by the assumption. The complement of the union $\cup Z_\sigma$ is also nowhere dense, since the set Σ is countable. Finally, for any $\lambda \in \mathbb{R}^n \setminus (\cup Z_\sigma)$, $F(A_\varphi)$ is a positive measure supported by Σ and the support of A_φ is contained in the u.d. set $\Lambda - \Lambda$. By Theorem 1 applied to A_φ , we conclude that Σ is a Γ -crystal with some lattice Γ . \square

4 Poisson structure of distributions with crystal geometry

Now we check that for any distribution as in Theorem 2, the Fourier transform can be calculated by means of the classical Poisson summation formula

$$F \left(\sum_{p \in L} \delta_p dx \right) = |\det X| \sum_{\sigma \in L^*} \delta_\sigma, \quad (4)$$

where L is a lattice in E , L^* is the dual lattice and $X = \{x_i(p_j)\}$ where p_1, \dots, p_n are generators of L . Equation (4) still holds for any shift of the lattice L with the corresponding exponential factor in the second term. For an arbitrary linear operator $P(x, D)$ that is a polynomial on both its arguments and for any vector $q \in E$, we have

$$F \left(P(x, D) \sum_{p \in L+q} \delta_p dx \right) = \exp(-j \langle \sigma, q \rangle) P(j^{-1} D_\sigma, j\sigma) \sum_{\sigma \in L^*} \delta_\sigma, \quad (5)$$

where we use notation $\langle \sigma, q \rangle$ for evaluation of the inner product of the covector σ and the vector q . Exponential factor $\exp(-j \langle \eta, x \rangle)$ inserted in the bracket causes the shift in the right hand side by the vector $\eta \in E^*$.

Theorem 5 *Any tempered distribution t supported by a crystal $\Lambda = \cup_1^I (L + q_i)$ in E such that \hat{t} is supported by a crystal $\Sigma = \cup_1^K (\Gamma + \eta_k)$ in E^* has the Poisson form*

$$t = \sum_{k=1}^K \sum_{i=1}^I \exp(-j \langle \eta_k, x \rangle) P_{ik}(x, D) \sum_{p \in L + q_i} \delta_p, \quad (6)$$

where $P_{ik}(x, D)$ are differential operators with polynomial coefficients. Moreover, $\Gamma = L^*$ and \hat{t} can be calculated by means of (5).

If t and \hat{t} are measures, all P_{ik} are constants and representation (6) was obtained in [14].

Proof. Let

$$G \doteq \{x \in E; |\langle g_i, x \rangle| \leq 1, i = 1, \dots, n\}, \quad (7)$$

where $g_1, \dots, g_n \in E^*$ are generators of the lattice L^* . For any $q \in L$, we denote $\Lambda(q) = \Lambda \cap (q + (k+1)KG)$, where $k = \text{ord } t$. Distribution

$$t[q] = \sum_{p \in \Lambda(q)} t_{p-q}(D) \delta_{p-q}$$

is supported by the compact set $\Lambda(0)$ and has order $\leq m$. The space Δ_m of all distributions supported on $\Lambda(0)$ of order $\leq m$ has finite dimension. Let e_1, \dots, e_n be the generators of L . For any $i = 1, \dots, n$, distributions $t[je_i]$, $j = 0, 1, \dots, J(i)$ belong to Δ_m and are linearly dependent, if $J(i)$ is sufficiently big. This yields

$$\sum_{j=1}^{J(i)} c_i^j t[je_i] = 0$$

for some constants c_i^j , $j = 1, \dots, J(i)$, which means that the sum

$$s_i \doteq \sum_j c_i^j t(x - je_i)$$

vanishes on $\Lambda(0)$. The Fourier transform \hat{v}_i is supported by Σ , since so does \hat{t} . This implies that $s_i = 0$ by Proposition 6 below, consequently t satisfies the system of difference equations

$$\sum_j c_i^j t(x - je_i) = 0, \quad i = 1, \dots, n. \quad (8)$$

According to the classical result, the general solution of the system (8) for a scalar function f on \mathbb{Z}^n has the form

$$f(x) = \sum_{k=1}^K \exp(\langle \lambda_k, x \rangle) p_k(x), \quad x \in \Lambda,$$

where $\lambda_k \in \mathbb{R}^n$, $k = 1, \dots, K$ are solutions of the characteristic system

$$\sum_{j=1}^n c_i^j \langle \lambda_k e_i \rangle^j = 0, \quad i = 1, \dots, n,$$

and p_k , $k = 1, \dots, K$ are some polynomials of degree $< m_k$, where m_k is the multiplicity of λ_k . The same formula holds for solutions t with values in an arbitrary vector space V . For $V = \Delta_m$, we obtain

$$t(x) = \sum_{k=1}^K \exp \langle \lambda_k x \rangle P_k(x, D) \delta_x, \quad x \in \Lambda,$$

where P_k are polynomials of $2n$ variables. Note that λ_k is pure imaginary if $P_k \neq 0$, since t is a tempered distribution and has at most polynomial growth at infinity. Therefore we can write $\lambda_k = -j\eta_k$ for a real vector η_k . This implies (5), where P_{ki} is the restriction of P_k to the crystal $L + q_i$, $i = 1, \dots, I$: \square

5 Gap theorem for crystals and the end of the proof

Proposition 6 *Let \hat{T} be a tempered distribution on E such that $F(T)$ is supported by a Γ -crystal $\Sigma = \cup_1^K (\Gamma + \eta_k)$ in E^* . If ν vanishes on a neighborhood of $(l+1)KG$, where G is as in (7) and $\text{ord } \hat{T} = l$, then $\nu = 0$.*

Proof. Let ψ be a test function on E^* with support in the ball $B(\rho)$ of small radius $\rho > 0$ such that $\psi(0) = 1$. For a number k , $1 \leq k \leq K$, we set

$$\Psi_k(\xi) = \psi(\xi) \prod_{j \neq k} \prod_{i=1}^n \sin^{m+1} \langle h_i, \xi - \eta_j \rangle \prod_{i=1}^n \text{sinc}^{m+1} \langle h_i, \xi - \eta_k \rangle, \quad (9)$$

where $\text{sinc} \lambda = \sin \lambda / \lambda$ and h_1, \dots, h_n are generators of Γ^* . We have $\nabla^j \Psi(\xi) = 0$ for any $j = 0, \dots, m$ and $\xi \in \Sigma \setminus \{\eta_k\}$, and $\nabla^j \Psi(\eta_k) = 0$, for $j = 1, \dots, m$, $\Psi(\eta_k) = 1$. It follows that $\hat{T}_0 = \hat{T}(\Psi_k) = T(F(\Psi_k))$. On the other hand,

$$\text{supp } F(\Psi_k) \subset (l+1)KQ + B(\rho),$$

if ρ is sufficiently small. This follows from $\widehat{\text{sinc}} = [-1, 1]$. By the assumption, $T(F(\Psi_k)) = 0$ which implies $\hat{T}_{\eta_k, 0} = 0$, since Ψ_k vanishes on Σ except the point $\eta_k = 0$. Replacing several factors $\text{sinc} \langle h_i, \xi - \eta_k \rangle$ in (9) by $\sin \langle h_i, \xi - \eta_k \rangle$, we apply the same arguments and check that $\hat{T}_{\eta_k, i} = 0$ for any i . This implies that $\hat{T}_{\eta_k} = 0$. The same arguments can be applied to any shift of vectors η_k , which gives $\hat{T}_\sigma = 0$ for any $\sigma \in \Gamma + \eta_k$ and any k . \square

End of the proof of Theorem 2. According to Lemma 4, tempered function $\hat{t} = F(t)$ is supported by the Γ -crystal Σ , whereas t is supported by a u.d. set Λ . Changing roles of E and E^* , we note set $\Sigma - \Sigma$ is u.d., since it is a Γ -crystal, the distribution $s \doteq \hat{t} d\xi$ is supported by Σ and distribution $t = F^{-1}(s) dx$ is supported by u.d. set Λ . By Lemma 4, Λ is a Δ -crystal with a lattice Δ in E . By Theorem 5, $\Gamma = \Delta^*$ which completes the proof. \square

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